Reciprocity of higher conserved charges in the $\mathfrak{sl}(2)$ sector of $\mathcal{N} = 4$ SYM

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September 29, 2010

Abstract

We extend the anlysis of the generalized Gribov-Lipatov reciprocity to the higher conserved charges of type IIB superstring on $AdS_5 \times S^5$. The property is shown to hold for twist $L = 2$, and 3 operators in the $\mathfrak{sl}(2)$ subsector.

1 Introduction and Discussion

In the last years, integrability emerged as a powerful tool in the investigation of the AdS/CFT correspondence. The integrable spin chain description of the dilatation operator led to the all-loop conjectured Bethe Ansatz equations for the $\mathfrak{psu}(2,2|4)$ algebra [1, 2, 3], completely describing (once supplemented with the dressing phase [4, 5]) the anomalous dimensions (and the full tower of conserved charges) of the model, up to wrapping effects. The presence of an infinite set of conserved charges q_k, forcing the factorizability of the scattering matrix for elementary excitations, is indeed a key manifestation of the integrability of a quantum model.

On the string side of the duality, the corresponding (classical) $\sigma-$model living on the string worldsheet in $AdS_5 \times S^5$ is also integrable, and the tower of non local conserved charges was derived in [6, 7, 8].

Despite the physical relevance of q_2, identified with the anomalous scaling dimension - string energy, the first charge does not play a special role from the point of view of the integrability, and all the charges are on equal footing. In [9] the weak-strong coupling correspondence of the full tower of charges in the $\mathfrak{su}(2)$ sector has been studied, but the physical meaning and properties of the higher conserved charges remains less understood.

In this work we investigate the reciprocity properties (see [10] for a review) of the first higher conserved charges in the $\mathfrak{sl}(2)$ sector. Reciprocity has its far origin in QCD in a symmetric treatment of the Deep Inelastic Scattering (DIS) and electron-positron into hadrons. The modified symmetric DGLAP kernel $P(N)$ in the evolution equation obeys the relation: $\gamma(N) = P\left(N + \frac{1}{2}\gamma(N)\right)$, where $\gamma(N)$ is the lowest anomalous dimension, and the reciprocity can be recast in the form of an asymptotic, large spin condition $P(N) = \sum_{\ell \geq 0} \frac{a_\ell(\log J^2)}{J^{2\,\ell}}$, where

$J^2 = N(N+1)$, a_ℓ are coupling-dependent polynomials and J^2 is the Casimir of the collinear subgroup $SL(2,\mathbb{R}) \subset SO(2,4)$. This condition can also be interpreted as a parity invariance $J \to -J$ in the large spin regime.

The $\mathfrak{sl}(2)$ sector, spanned by single trace operators $\mathcal{O} \sim \mathrm{Tr}\,(\mathcal{D}^{n_1} Z \ldots \mathcal{D}^{n_L} Z)$, is a closed subsector of the theory under perturbative renormalization; $N = \sum n_i$ is the total spin and L is the classical dimension minus the spin (twist) of the operator. The relevant dual string state is the classical folded (S, J) string solution, describing a string extended in the radial direction of AdS_5 and rotating in AdS_5, with center of mass moving on a circle of S^5 [11, 12]. This solution is linked via analytic continuation $E \to -J_1,\ S \to J_2,\ J \to -E$ to the string configuration (J_1, J_2) with two angular momenta on S^5, for which the higher charges at strong coupling have been constructed in [9] by using the Bäcklund transformations of the integrable classical string σ-model.

Analysing the first two charges at weak coupling $q_{4,6}$ (respectively at 3-loop plus 4-loop dressing part and 2-loop level) and the first charges at strong coupling at classical level, we find that the reciprocity condition can be consistently generalized for all the tested higher charges.

2 Higher Charges and Reciprocity at Weak Coupling

In the weak coupling regime the closed formulae for multi-loops higher charges can be efficiently computed following the Baxter approach [13, 14, 15] toghether with the maximum transcendentality Ansatz (and then completed by the dressing factor starting from the four-loop order), resulting in a combination of harmonic sums of definite transcendentality [16]. The reciprocity condition for the full tower of conserved charges can be generalized from the condition for q_2 defining the kernel $P_r(N)$ as

$$q_r(N) = P_r\left(N + \frac{1}{2}\, q_2(N)\right). \qquad (1)$$

This equation emphasizes the role of the renormalized conformal spin, as also suggested by light cone quantization. Reciprocity implies a constraint on the form of the expansion of P_r at large N, which should involve only integer inverse powers of $N(N+1)$. The check of this property is easier after a rewriting of the charges in terms of the Ω basis [17], where the reciprocity simply means that the Ω must have odd positive or even negative indices. We report here only the first, parity respecting results for the higher charge q_4:

$L = 2$, **three-loops reciprocity of** q_4

$$\begin{aligned}
P_4^{(1)} &= 16\,(\Omega_3 + 6\Omega_{-2,1})\,, & (2)\\
P_4^{(2)} &= -\frac{16}{5}(\pi^4\Omega_1 + 120\Omega_{-4,1} + 20\pi^2\Omega_{-2,1} + 60\Omega_{-2,3} + 60\Omega_{1,-4} + 20\pi^2\Omega_{1,-2} \\
&\quad +120\Omega_{-2,1,-2} + 120\Omega_{1,-2,-2} - 480\Omega_{1,-2,1,1}), & (3)\\
P_4^{(3)} &= \frac{32}{15}(180\zeta(3)\Omega_{-4} + 2\pi^6\Omega_1 + 3\pi^4\Omega_3 - 30\pi^2\Omega_5 - 720\Omega_7 + 900\Omega_{-6,1} + 240\pi^2\Omega_{-4,1} \\
&\quad +540\Omega_{-4,3} + 30\pi^4\Omega_{-2,1} + 60\pi^2\Omega_{-2,3} + 720\Omega_{1,-6} + 240\pi^2\Omega_{1,-4} + 36\pi^4\Omega_{1,-2} \\
&\quad +180\Omega_{3,-4} + 60\pi^2\Omega_{3,-2} - 180\Omega_{5,-2} + 2520\Omega_{-4,-2,1} + 2160\Omega_{-4,1,-2}
\end{aligned}$$

$$
\begin{aligned}
&+1080\Omega_{-2,-4,1}+360\Omega_{-2,-2,3}+1800\Omega_{-2,1,-4}+120\pi^2\Omega_{-2,1,-2}\\
&+1080\Omega_{-2,3,-2}+1440\Omega_{1,-4,-2}+2160\Omega_{1,-2,-4}\\
&+240\pi^2\Omega_{1,-2,-2}+1440\Omega_{1,1,5}+2160\Omega_{1,5,1}+360\Omega_{3,-2,-2}+720\Omega_{5,1,1}\\
&-1440\Omega_{-4,1,1,1}+2160\Omega_{-2,-2,-2,1}\\
&+1440\Omega_{-2,-2,1,-2}+720\Omega_{-2,1,-2,-2}-2880\Omega_{1,-4,1,1}\\
&+1440\Omega_{1,-2,-2,-2}-960\pi^2\Omega_{1,-2,1,1}-1440\Omega_{1,-2,1,3}-1440\Omega_{1,-2,3,1}-1440\Omega_{1,1,-4,1}\\
&-960\pi^2\Omega_{1,1,-2,1}-1440\Omega_{1,1,-2,3}-1440\Omega_{3,-2,1,1}-2880\Omega_{-2,-2,1,1,1}\\
&-2880\Omega_{-2,1,-2,1,1}-5760\Omega_{-2,1,1,-2,1}-2880\Omega_{-2,1,1,1,-2}-2880\Omega_{1,-2,-2,1,1}\\
&-5760\Omega_{1,-2,1,-2,1}-5760\Omega_{1,-2,1,1,-2}-11520\Omega_{1,1,-2,-2,1}\\
&-5760\Omega_{1,1,-2,1,-2}+11520\Omega_{1,1,-2,1,1,1}+360\Omega_{1,1}\zeta(5)-240\pi^2\Omega_{1,1}\zeta(3)\\
&-720\Omega_{-2,1,1}\zeta(3)-720\Omega_{1,-2,1}\zeta(3)-720\Omega_{1,1,-2}\zeta(3)) \qquad (4)
\end{aligned}
$$

$L=2$**, four-loops reciprocity of the dressing part of** q_4

$$
\begin{aligned}
P_4^{(4,\text{dressing})} \quad = \quad & 3072\Omega_{-6}+3072\Omega_{-2,-4}+3072\Omega_{5,1}-18432\Omega_{-4,1,1}\\
&-12288\Omega_{-2,1,3}-12288\Omega_{-2,3,1}-6144\Omega_{1,-4,1}-6144\Omega_{1,-2,3}\\
&-24576\Omega_{-2,-2,1,1}-12288\Omega_{-2,1,-2,1}-12288\Omega_{1,-2,-2,1}\\
&+98304\Omega_{-2,1,1,1,1}+24576\Omega_{1,-2,1,1,1}. \qquad (5)
\end{aligned}
$$

3 Higher Charges and Reciprocity at Strong Coupling

The string state dual of the gauge operators is the semiclassical $\mathfrak{sl}(2)$ folded string. As anticipated, it is related to the (J_1, J_2) string by an analytic continuation, mapping one into another the σ-models describing the strings on $AdS_3\times S^1$ and $R\times S^3$, as well as the relative equations of motion, their solutions and the conserved charges. Energy $\mathcal{E}=E/\sqrt{\lambda}$, spin $\mathcal{S}=S/\sqrt{\lambda}$, and angular momentum $\mathcal{J}=J/\sqrt{\lambda}$ for the folded string are related by

$$
\sqrt{\kappa^2-\mathcal{J}^2}=\frac{1}{\sqrt{\eta}}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2},1,-\frac{1}{\eta}\right),\quad \omega^2-\mathcal{J}^2=(1+\eta)\,(\kappa^2-\mathcal{J}^2), \qquad (6)
$$

$$
\mathcal{S}=\frac{\omega}{\sqrt{\kappa^2-\mathcal{J}^2}}\,\frac{1}{2\,\eta\,\sqrt{\eta}}\,{}_2F_1\left(\frac{1}{2},\frac{3}{2},2,-\frac{1}{\eta}\right),\quad \mathcal{E}=\frac{\kappa}{\sqrt{\kappa^2-\mathcal{J}^2}}\,\frac{1}{\sqrt{\eta}}\,{}_2F_1\left(-\frac{1}{2},\frac{1}{2},1,-\frac{1}{\eta}\right) \qquad (7)
$$

and for the comparison with the gauge theory results we are interested in the slow string limit; using $\mathcal{J}$ as an expansion parameter, the quantum contribution to the energy can be computed as:

$$
\eta(\mathcal{S},\mathcal{J})=\eta^{(0)}(\mathcal{S})+\eta^{(2)}(\mathcal{S})\,\mathcal{J}^2+\eta^{(4)}(\mathcal{S})\,\mathcal{J}^4+\cdots\ , \qquad (8)
$$

$$
\Delta=\mathcal{E}-\mathcal{S}=\Delta^{(0)}(\mathcal{S})+\Delta^{(2)}(\mathcal{S})\,\mathcal{J}^2+\Delta^{(4)}(\mathcal{S})\,\mathcal{J}^4+\cdots\ . \qquad (9)
$$

Introducing the function f defined as $\Delta(\mathcal{S})=\mathcal{E}(\mathcal{S})-\mathcal{S}=f\left(\mathcal{S}+\frac{1}{2}\,\mathcal{E}(\mathcal{S})\right)$ and expanding perturbatively, reciprocity is translated in the absence of inverse odd powers of $\mathcal{S}$ in the expansions.

Higher charges $\mathcal{E}_{4,6,\ldots}$ can be constructed in the $\mathfrak{su}(2)$ sector by using the Bäcklund transformation method [9], and then analytically continued ($t \to -1/\eta$, $\mathcal{E}_2 \to J$). As an example, for the first non-vanishing charge $\mathcal{E}_4$ we get:

$$\mathcal{E}_4 = -\tfrac{16}{\pi^2\,\mathcal{E}_2}\,Z_1(t) + \tfrac{32}{\pi^4\,\mathcal{E}_2^3}\,Z_2(t),$$
$$Z_1(t) = \mathbb{K}(t)[\mathbb{E}(t) + (t-1)\mathbb{K}(t)], \qquad Z_2(t) = t(t-1)\mathbb{K}(t)^4 \tag{10}$$

where t is a modular parameter. In analogy with the case of the energy, we propose to test reciprocity on the functions f_k defined by

$$Z_k(\mathcal{S}) = f_k\left(\mathcal{S} + \frac{1}{2}\mathcal{E}(\mathcal{S})\right), \quad Z_k(\mathcal{S}) \equiv Z_k(-1/\eta(\mathcal{S})). \tag{11}$$

Using the Lagrange-Bürmann formula [18]

$$f(\mathcal{S}) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{d}{d\mathcal{S}}\right)^{k-1}\left[\left(-\frac{\Delta(\mathcal{S})}{2}\right)^k Z'(\mathcal{S})\right] = Z(\mathcal{S}) - \frac{1}{2}\,\Delta(\mathcal{S})\,Z'(\mathcal{S}) + \cdots \tag{12}$$

from $\eta = \eta(\mathcal{S}, \mathcal{J})$ we obtain an expansion for $f_k(\mathcal{S}) = f_k^{(0)}(\mathcal{S}) + f_k^{(2)}(\mathcal{S})\,\mathcal{J}^2 + f_k^{(4)}(\mathcal{S})\,\mathcal{J}^4 + \cdots$ and computing 0-th order correction for Z_1 and Z_2 we find the result

$$\begin{aligned} f_1^{(0)} &= -\frac{1}{4}\left(\log\bar{\mathcal{S}} - 2\right)\log\bar{\mathcal{S}} + \boxed{0\cdot\frac{1}{\bar{\mathcal{S}}}} + 2\left(2 - 3\log\bar{\mathcal{S}}\right)\log\bar{\mathcal{S}}\,\frac{1}{\bar{\mathcal{S}}^2} + \boxed{0\cdot\frac{1}{\bar{\mathcal{S}}^3}} \\ &\quad + \left(80\log^3\bar{\mathcal{S}} - 118\log^2\bar{\mathcal{S}} + 23\log\bar{\mathcal{S}} + 1\right)\,\frac{1}{\bar{\mathcal{S}}^4} + \boxed{0\cdot\frac{1}{\bar{\mathcal{S}}^5}} + \cdots, \end{aligned} \tag{13}$$

$$\begin{aligned} f_2^{(0)} &= \frac{1}{16}\log^4\bar{\mathcal{S}} + \boxed{0\cdot\frac{1}{\bar{\mathcal{S}}}} + \log^4\bar{\mathcal{S}}\,\frac{1}{\bar{\mathcal{S}}^2} + \boxed{0\cdot\frac{1}{\bar{\mathcal{S}}^3}} \\ &\quad -\frac{1}{2}\left(\log^3\bar{\mathcal{S}}\left(16\log^2\bar{\mathcal{S}} - 22\log\bar{\mathcal{S}} - 1\right)\right)\,\frac{1}{\bar{\mathcal{S}}^4} + \boxed{0\cdot\frac{1}{\bar{\mathcal{S}}^5}}, \end{aligned} \tag{14}$$

where the absence of inverse odd powers of $\mathcal{S}$, highlighted by the boxes, clearly supports parity invariance. The procedure can be straightforwardly extended to the next conserved charges, showing parity invariance in all the tested cases.

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